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Nonstationary Quantum Mechanics. III. Quantum Mechanics Does Not Incorporate Classical Physics

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It is shown that disagreement between the prediction of classical and conventional quantum mechanics about momentum probabilities exists in the case of a quasiclassical motion. The discussion is based on the detailed consideration of two specific potentials: $U(x)=x$ and the oscillatory potential $U(x)=m\omega^2x^2/2$. The results of the present Part III represent a further development of the idea in Todorov (1980) about the possible inefficiency of conventional theory in the case of potentials swiftly varying with time.

In Todorov (1980) (hereafter referred to as T1) we argued that the evolution equation for quantum states of particles is asymmetric in respect to time reversal and that it differs essentially from the Schrödinger equation (SE) in the presence of potentials swiftly varying with time. This can lead to disagreement between theory and experiments in such cases. The experimental results discussed in T1 seem to support this statement. Besides, the expected time irreversibility of the basic evolution equation provides a natural explanation of the entropy increase with time which cannot be achieved with the help of the time-reversible SE for a subsystem of a larger system—cf. Todorov (1980) (hereafter referred to as T2).

We shall undertake here another step in the direction outlined in our cited work. Our aim is to show with the help of simple mathematics that the nonstationary SE can be inefficient in providing some expected results. More precisely, we shall show that discrepancies exist between expected probabilities and probabilities calculated with the help of the nonstationary SE exactly in the range of momenta (and kinetic energies) where they are least of all suspected.

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The above statement and terminology imply that we shall concentrate our attention on the case of quasiclassical motion of particles. We know the exact meaning of the words "momentum" and "kinetic energy" and the way to calculate the corresponding distributions in this case (for unspecified initial conditions about the state of motion). We know, besides, in what type of experiments one can find the said distributions empirically. We also know that in its range of validity (large, though not relativistic energies of motion) Newtonian mechanics is true, so that the SE must reproduce the classical results for large energies of motion since it is meant to be a nonrelativistic generalization of the classical equations of motion.

In what follows we shall examine specific distributions from the classical and conventional quantum mechanical point of view.

Consider the one-dimensional stationary SE. It was pointed out by Einstein (1953) that it yields a statistical "snapshot" of the x distribution in a large ensemble of particles in the same eigenstate ψ_n and this distribution will coincide with the classical result when n is large enough after an averaging of $|\psi_n|^2$ over a suitable region containing many oscillations of the wave function (WF) (Einstein had in mind the simple example of a particle in an infinitely deep potential well, but the above has a general validity for all cases of a quasiclassical motion—see below). In the (quasi)classical case the location of a particle determines precisely its kinetic energy and the modulus of its momentum when the total energy of motion and the course of the potential energy are known. We shall have thus the following equation for a classical ensemble of identical systems with random initial conditions of motion in the case of a potential energy, symmetrical in respect to point $x=0$ and monotonically increasing to the right and to the left of this point.

$$w(x)\,dx = w(p)\,dp \tag{1}$$

where $w(x)$ and $w(p)$ are the corresponding densities of probabilities for specific values of the coordinate and the momentum of the particle; the differentials dx and dp, certainly, are not independent. In the case when $U(x)=\infty$ for $x\leqslant 0$ and $U(x)$ monotonically increases to the right of the origin $x=0$, starting from a fixed value [e.g., $U(x)=0$] we shall have

$$w(x)\,dx = 2w(p)\,dp, \qquad p>0 \tag{2}$$

[let us recall that the momentum has two possible signs so that in our case $w(p)=w(-p), p>0$]. The function $w(p)$ can be determined experimentally by a (quasi)instantaneous "switching off" of the potential $U(x)$ and a subsequent measurement of the momentum distribution in a sufficiently large ensemble of independent identical systems. The results which the nonstationary SE will give for this case should coincide with the classical ones due to the validity of the classical picture. We shall check below whether this is really so.

We shall consider in detail two specific examples of potentials which belong to the above types: (i) $U(x)=\infty$ for $x \leqslant 0$, $U(x)=x$ for $x>0$ (the corresponding constant factor before x is chosen to be $=1$), and (ii) $U(x)=m\omega^2x^2/2$ (the usual potential energy for a particle of mass m oscillating with a frequency ω). Applying the well-known condition $p^2 \gg \hbar|dp/dx|$ for validity of the quasiclassical approximation one can easily see that in both cases $\lim_{n\to\infty}(\Delta_n/A_n)=0$, where Δ_n is the interval along the x axis in which the said approximation is not valid and A_n is the classical amplitude of motion along x of a particle of energy E_n, E_n being the nth eigenvalue of the quantum Hamiltonian H. In such a way we demand that classical mechanics and the nonstationary SE give identical results for $w(p)$ with the exception of the range of small momenta (large x) where the classical picture is known to be inadequate from the results of the stationary SE.

It is worth recalling that in the regions where the quasiclassical approximation is valid the WF $\psi(x)$ is equal to

$$\psi(x)=\frac{C}{[p(x)]^{1/2}}\cos\left(\frac{1}{\hbar}\int_b^x p\,dx+\frac{\pi}{4}\right)=\frac{C}{[p(x)]^{1/2}}\sin\left(\frac{1}{\hbar}\int_x^b p\,dx+\frac{\pi}{4}\right) \tag{3}$$

where b is one of the reversal points in classical mechanics [$U(x)>E$ for $x>b$, $U(x)\leqslant E$ for $a\leqslant x\leqslant b$] and C is constant. The meaning of the averaging procedure $\langle|\psi^2|\rangle$ in such regions is transparent now: The condition

$$\left|\hbar\frac{dp}{dx}/p^2\right|=\left|\frac{d\lambda\!\!\!^-}{dx}\right|\ll 1$$

shows that the amplitude $C^2/p(x)$ of $\psi^2(x)$ varies slowly over distances $\sim\lambda(x)$ so that it is practically constant when one takes the value $\langle|\psi^2|\rangle$ over a region containing a suitable number of oscillations of $\psi(x)$ and only $\cos^2$ remains under the symbol of averaging. The said slow variation of $p(x)$ makes the taking of the average value of $\cos^2$ over the same region sensible. The proportionality of $|\psi^2|$ to $p(x)^{-1}$ is also clear from a classical point of view—the probability of finding a particle in a small interval about point x is inversely proportional to $v(x)$. Thus classical mechanics is applicable exactly in those regions where the above procedure is sensible.

In the regions in which one must apply quantum mechanics both $p(x)^{-1}$ and $\int_b^x p\,dx$ behave in an "irregular" fashion which precludes the possibility of a reasonable averaging procedure. In such districts one has to return to the coordinate distribution given by $|\psi|^2$. Because of the essential departure of the quantum equation of motion from the classical one we must not expect even coincidence of the overall quantum and classical probabilities to find the particle in a region containing several irregular oscillations of the WF.

Let us examine now the above two cases. They are well known from the courses of quantum mechanics. Taking the first case we come to the following equation for the eigenvalues of H:

$$\frac{d^2\psi_n}{dx^2}-\frac{2m}{\hbar^2}(x-E_n)\psi_n=0 \tag{4}$$

The solution of (4) with the property $\psi_n(x)\to 0$ for $x\to\infty$ is

$$\psi_n(x)\simeq\int_{-\infty}^{\infty}\exp\left[i\left(\frac{p^3}{6m}-E_n p+px\right)/\hbar\right]dp, \qquad x\geqslant 0 \tag{5}$$

where the eigenvalues $E_n>0$ are determined by the requirement $\psi_n(0)=0$ $[U(x)=\infty$ for $x\leqslant 0]$ and $\simeq$ denotes proportionality (with $\sim$ we shall denote further the order of magnitude).

Having in mind the asymptotic properties of the Airy functions [cf., e.g., Smirnov 1969, Section 118] we see that $\psi_n(x)$ tends exponentially to zero for $x>E_n$ while in the classical region we have $\langle|\psi|^2\rangle\simeq(E_n-x)^{-1/2}$ in exact correspondence with the prediction of classical mechanics about $w_n(x)$ for a particle of energy E_n (the averaging is carried out in accord with the above; let us recall here that we work with very large n and, correspondingly, E_n). This result, obtained with the help of the stationary SE, is a concrete expression of the fact that the use of this equation is reasonable and justified by experiment.

It is readily seen that classical mechanics gives $w_n(p)=\text{const}$ when $p^2/2m\leqslant E_n$, and $w_n(p)=0$ for $p^2/2m>E_n$. Let us see what is the prediction of the nonstationary SE in this case.

The density of probability of finding a value p' of the momentum is given, as is well known, by $|a_n(p')|^2$, where

$$a_n(p')=\frac{1}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{\infty}\psi_n(x)e^{-ip'x/\hbar}\,dx \tag{6}$$

In order to obtain $a_n(p')$ from the integral representation (5) of $\psi_n(x)$, $n\to\infty$, we have to integrate first over x (from 0 to ∞). This becomes possible by using the customary trick of multiplying $\psi_n(x)$ by a factor $\exp[-\alpha x/\hbar]$ and letting α tend to $+0$ after the integration (this is certainly admissible in our specific case). It immediately follows from (5) that

$$a_n(p') \simeq \int_0^{\infty}\int_{-\infty}^{\infty} \exp\left[\frac{1}{\hbar}\left(\frac{p^3}{6m} - E_n p\right)\right] \exp\left[\frac{i}{\hbar}(p-p'+i\alpha)x\right] dp\, dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \frac{\exp\left[\frac{i}{\hbar}\left(\frac{p^3}{6m} - E_n p\right)\right]}{p-p'+i\alpha} = -\pi\hbar \exp\left[\frac{i}{\hbar}\left(\frac{p'^3}{6m} - E_n p'\right)\right]$$

$$-i\hbar \fint_{-\infty}^{\infty} \frac{\exp\left\{i\left[\frac{(p+p')^3}{6m} - E_n(p+p')\right]/\hbar\right\}}{p} dp \tag{7}$$

where $\fint$ denotes the principal value of the integral and the well-known formula

$$\lim_{\alpha \to +0} \frac{1}{p+i\alpha} = -i\pi\delta(p) + P\frac{1}{p}$$

is used.

After some elementary manipulations the second term on the right-hand side of the last equality (7) is transformed into

$$2\hbar \exp\left[\frac{i}{\hbar}\left(\frac{p'^3}{6m} - E_n p'\right)\right] \int_0^{\infty} \frac{1}{p} \sin\left(\frac{p^3}{6m\hbar} + \frac{pp'^2}{2m\hbar} - \frac{E_n p}{\hbar}\right) \exp\left[\frac{i}{\hbar}\frac{p'p^2}{2m}\right] dp$$

Expanding sin(...) in the usual way

$$\sin\left[\frac{1}{\hbar}\left(\frac{p^3}{6m} + \frac{pp'^2}{2m} - E_n p\right)\right] = \sin\frac{p^3}{6m\hbar}\cos\frac{p}{\hbar}\left(\frac{p'^2}{2m} - E_n\right)$$

$$+ \sin\frac{p}{\hbar}\left(\frac{p'^2}{2m} - E_n\right)\cos\frac{p^3}{6m\hbar} \tag{8}$$

and having in mind that the first term on the right-hand side of (8) oscillates very fast when $|E_n - p'^2/2m|/\hbar \gg 1$ in any case and that

$$\lim_{L \to \infty} \frac{\sin Lp}{p} = \pi\delta(p)$$

so that

$$\int_0^\infty \frac{\sin Lp}{p} f(p)\,dp = \frac{\pi}{2} f(0), \qquad L \to \infty \tag{9}$$

we obtain in our case $E_n \to \infty$ the final result

$$a_n(p') \simeq -2\pi\hbar \exp\left[\frac{i}{\hbar}\left(\frac{p'^3}{6m} - E_n p'\right)\right] \tag{10}$$

for all p' satisfying the inequalities $p'^2/2m < E_n$,

$$\frac{1}{\hbar}\left|\frac{p'^2}{2m} - E_n\right| \gg 1$$

and

$$a_n(p') = 0 \tag{11}$$

for

$$p'^2/2m > E_n, \qquad \left|\frac{1}{\hbar}\left(\frac{p'^2}{2m} - E_n\right)\right| \gg 1$$

Having in mind that E_n is very large this means that (10) and (11) will reproduce the classical results for $w_n(p')$ "almost everywhere," the exception being the region of kinetic energies (denoted by δ) in which $(1/\hbar)|p'^2/2m - E_n|$ is, roughly speaking, of the order of magnitude of unity. In this range of kinetic energies $|a_n(p')|^2$ is transformed into zero from its practically constant initial value and the classical result is not reproduced there.

But the region in which $p'^2/2m \approx E_n$, $p'^2/2m < E_n$, corresponds to an interval $(0, \Delta)$ along the x axis in which the motion of the particle is "most classical" ($|p^2|\hbar\, dp/dx|$ has the largest value there). In other words, the disagreement between the nonstationary SE and classical mechanics exists exactly in the range where we have the greatest reason to require a reproduction of the classical results. On the other hand this equation unneccessarily precisely reproduces the classical result for $w_n(p')$ in the region of small p' ($x \approx E_n$) where the motion of the particle is known not to be classical. A possible explanation of this nonphysical fact about the momentum distribution can be found in the Appendix.

The length of the energy interval δ, obviously, remains constant irrespective of what large E_n we choose and the same applies to the corresponding interval in coordinate space to the right of $x=0$ [where $U(\Delta)-U(0)=\delta$]. Thus the said disagreement between classical mechanics and the nonstationary SE cannot be removed in a coordinate (and energy) interval which may be treated as macroscopic in the case of very large E_n since the number of oscillations of the WF in this fixed interval along the x axis tends to infinity together with E_n.

It is interesting to note that in the case $U(x)=x$, $-\infty<x<\infty$ (unbounded motion with a continuous energy spectrum) the predictions of classical mechanics coincide with those of the nonstationary SE. The mechanism of this phenomenon in the said case can be found in the fact that the region in which no agreement exists has gone to $-\infty$.

One may argue here that the classical picture is gradually restored, in a sense, for $E_n\to\infty$ since the interval Δp in momentum space in which $w_n(p)\neq|a_n(p)|^2$ tends to zero with the increase of E_n. This objection, however, is easily rejected having in mind the above constancy of δ and $(0,\Delta)$, which are macroscopic. But it is certainly worth examining a specific example in which Δp does not tend to zero and in which the nonstationary SE leads to the same paradoxical results.

Consider the case of an oscillatory potential energy $U(x)=m\omega^2x^2/2$. In the classical consideration equation (1) will be valid. The quantum results for E_n and ψ_n are well known:

$$\psi_n(x)=\frac{1}{\pi^{1/4}}\frac{1}{(2^n n!)^{1/2}}e^{-x^2/2}H_n(x) \tag{12}$$

$$E_n=\hbar\omega\left(n+\tfrac{1}{2}\right) \tag{13}$$

(for the sake of convenience we have chosen parameters for which $m\omega/\hbar=1$ is fulfilled). The classical amplitude of motion A_n along x (for a specific E_n) is determined by $m\omega^2A_n^2/2=E_n$ and we have

$$w_n(x)=\left[\pi A_n\left(1-x^2/A_n^2\right)^{1/2}\right]^{-1} \tag{14}$$

for the classical coordinate density of probability. Having in mind that the classical velocity $v_n(x)=A_n\omega(1-x^2/A_n^2)^{1/2}$ and defining formally a classical "wave number" $k_n(x)$ as $p_n(x)/\hbar$, where $p_n(x)$ is the classical momentum in point x for a total energy E_n, we obtain immediately

$$k_n(x)=A_n\left(1-x^2/A_n^2\right)^{1/2} \tag{15}$$

so that, replacing p with k in equation (1) we shall have

$$w_n(k)=\left[\pi A_n\left(1-k^2/A_n^2\right)^{1/2}\right]^{-1} \tag{16}$$

for $k \leqslant A_n$, and

$$w_n(k)=0 \tag{17}$$

for $k>A_n$. In such a way we have a remarkable symmetry of the classical expressions (14) and (16).

That stationary SE will, certainly, reproduce the classical results for $w_n(x)$ after a corresponding averaging (see above) of $|\psi_n(x)|^2$, $n\to\infty$, with the exception of the region where

$$p^2/\hbar\left|\frac{dp}{dx}\right|\lesssim 1$$

[small p and large x ($x\approx A_n$)]. This can be easily checked near $x=0$ where the picture most of all resembles the classical one for $n=2l$, l being a large positive integer, keeping in mind that $A_n=(2n+1)^{1/2}$, $\psi_n(0)=(-2)^{n/2}1\cdot 3\cdot 5\ldots(n-1)$ and that $\psi_n(x)\approx\psi_n(0)\cos(2n+1)^{1/2}x$ for large even n and small x (Smirnov, 1969, Section 163).

An instantaneous exclusion of the force field will give the following values of $a_n(k)$ [$|a_n(k)|^2$ yielding the probability density for a value k of the wave number] according to the nonstationary SE:

$$a_n(k)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-ikx}\psi_n(x)\,dx=\frac{i^n}{\pi^{1/4}(2^n n!)^{1/2}}e^{-k^2/2}H_n(k) \tag{18}$$

(cf., e.g., Vilenkin, 1965), i.e., we have again a complete symmetry of the expressions for $|\psi_n(x)|^2$ and $|a_n(k)|^2$ as in the classical case. In order to obtain the classical result we have to take the average of $|a_n(k)|^2$ in exactly the same way as the one applied in finding $\langle|\psi_n(x)|^2\rangle$ {in the case $U(x)=x$ this specific averaging of $|a(k)|^2$ was not necessary; we must recall here that in any region where $\psi(x)$ [or $a(k)$] vary in an irregular fashion one should return to the usual densities $|a(k)|^2$ and $|\psi(x)|^2$ since, obviously, any averaging would be senseless in this case}. In such a way the function $P_n(k)=\langle|a_n(k)|^2\rangle$ will be an exact copy of the function $P_n(x)=\langle|\psi_n(x)|^2\rangle$ because of equations (12) and (18), the latter function reproducing the classical result everywhere with the exception of the above-mentioned region of large x ($x\sim A_n$), where the course of ψ_n is irregular. But then $P_n(k)$ will reproduce in the same way the classical result for $w_n(k)$ [equations (16) and (17)] with the exception of the regions of exactly the same length (due to the choice $m\omega/\hbar=1$) about $k_{n\max}=(2mE_n)^{1/2}/\hbar=A_n$, i.e., we shall have disagreement with classical mechanics exactly for such (large) values of k [and most slowly varying $U(x)$ correspondingly] for which agreement with the classical picture is obligatory. This is analogous to the result for $U(x)=x$ with the exception that the length of the corresponding region in the k space increases together with E_n in exactly the same way as the region in x space in which $p^2/\hbar|dp/dx|\sim 1$. As in the first case, we have a nonphysical reproduction of $w_n(k)$ for small k (≈ 0) which is obvious from our preceding discussion about the value of $\langle|\psi_n(x)|^2\rangle$ near $x=0$.

One may suspect that the presence of wave numbers $k>k_{n\max}$ predicted by the nonstationary SE and the disagreement between classical mechanics and this equation for $k\sim k_{n\max}$ (but smaller than $k_{n\max}$) has something to do with the nonclassical character of the motion for $x\sim A_n$. It is obvious, however, that only the regions (along x) in which $\psi_n(x)$ has approximately the same period of spatial oscillations as $\exp[-ikx]$ have a noticeable contribution to the value of $a_n(k)$ so that the region of large x ($x\sim A_n$) can be simply cut off when we are interested in large k. The presence of nonzero $a_n(k)$, $k>k_{n\max}$, is due to the smaller number of particles with $k\sim A_n$, $k<A_n$, than the one required by classical mechanics. So, the mechanism of the presence of such $a_n(k)$ is local and lies in the fact that the period of oscillations of $\exp[-ikx]$ for $k\approx A_n$, $k>A_n$, is approximately equal to the one of $\psi_n(x)$ near $x=0$. But a much more important fact is the quite different general behavior of $P_n(k)$ from the classical one expressed by $w_n(k)$ for $k\sim k_{n\max}$, $k<k_{n\max}$, in a region substantially larger than the one for $k>k_{n\max}$ in which $a_n(k)$ differs noticeably from zero in our specific case $U(x)\simeq x^2$. Indeed, $\psi_n(x)$ exponentially decreases when $x>A_n$, the same applying, obviously, to $a_n(k)$ for $k>A_n$. It is this general behavior of $P_n(k)$ exactly which makes it possible to say that the nonstationary SE and classical mechanics disagree in macroscopic regions of classical motion in this concrete case.

The above "local resonance" mechanism of nonclassical behavior of $|a_n(k)|^2$ in the neighborhood of $k_{n\max}$ is essentially influenced by the way in which the amplitude of $\psi_n(x)$ varies with x in the region of most classical motion. This will determine the length of the region in p space in which discord between classical and quantum distributions exists [compare the results for $U(x)\simeq x$ and $U(x)\simeq x^2$]. The said mechanism is encountered in its simplest form in the case of a potential well of the type $U(x)=\infty$ for $x\leqslant 0$, $U(x)=\infty$ for $x\geqslant a$, $U(x)=0$ inside the interval $0<x<a$. Let the walls of this well disappear instantaneously in moment

$t=0$. We have $\psi_n(x)=(2/a)^{1/2}\sin(n\pi x/a)$ inside the well and $\psi_n(x)=0$ outside it, $n=1,2,\ldots$, so that one immediately obtains in the case, e.g., of even n that $|a_n(|k|)|^2 \simeq n^2\pi^2 a \sin^2(ka/2)|(n^2\pi^2-k^2a^2)^2$ and when $n\to\infty$ one can easily see that $|a_n(|k|)|^2$ represents a peak with fast-vanishing pulsations about it. The half-width ($\sim 1/a$) and height ($\sim a$) remain practically constant with n. The center of the peak lies, certainly, in $k_{n\max}=n\pi/a$ (the only possible classical $|k|$ corresponding to the eigenenergy $E_n=n^2\pi^2\hbar^2/2ma^2$). In this specific case the motion of the particle is quasiclassical in the entire region $0<x<a$ (prior to the falling of the walls) but the above local mechanism makes it impossible to decrease the half-width $\sim 1/a$ irrespective of how large n may be. This constancy of $\Delta k \sim 1/a$ makes the present case intermediate compared to the two cases examined above in which Δk tends correspondingly to 0 and to ∞ with the increase of E_n. We have here, evidently, simply an illustration of the uncertainty relation $\Delta p \Delta x \sim \hbar$ ($\Delta x=a, \Delta p \sim \hbar/a$). Thus the uncertainty relation and the two cases discussed above have a common "resonance" mechanism, the said relation being its simplest manifestation. Only the consideration of cases more complex than simple potential wells can clearly demonstrate the paradoxical character of the resonance phenomenon in the classical limit since in such cases we have regions in which motion may be called "most classical" and in which disagreement appears between the nonstatic SE and classical mechanics while in the well case motion is equally classical everywhere (for large n).

It is well known that classical mechanics is a logically consistent theory which is postulated to be correct in its domain of validity discussed above. In the above examples we came to a contradiction between classical mechanics and the nonstationary SE in the domain where classical mechanics is accepted as correct. But we have reasons to assert that the application of the said equation is contradictory from the point of view of logic (T1). According to the discussion there we need something more than simply a WF $\psi(x,t)$ in order to obtain a correct theory in which evolution is described with the help of some master equation. Something more than a WF is called "hidden variables" in the accepted terminology. Our present discussion shows that the simplest HV theory—classical mechanics—should be preferred in the description of the results in nonstationary situations of the type discussed in detail above.

The above disagreement between the nonstationary SE and classical mechanics is in fact disagreement between the nonstationary and the stationary SEs: The fact that the static SE reproduces the classical results about the x distribution near $x=0$ is at the same time evidence about the validity of the classical momentum distribution picture since, as it was already said, classical coordinate distribution is obtained with the help of classical momentum distribution. We saw, besides, that the two equations give different implications about the motion of particles in the range of small momenta where the classical picture is not valid.

The fact that the regions in which disagreement between classical mechanics and the nonstatic SE exists are comparatively small may probably be explained with the existence of two contradictory tendencies: The tendency of a correct description of the classical case by quantum mechanics and the tendency of the SE to be incorrect in the nonstationary case (let us recall that we have in mind everywhere the SE with a variable external potential [$U(x, t)$ (T1)]). In the case of substantially quantum phenomena the disagreement between the nonstatic SE and experiment may be essentially stronger as implied by some experimental results (T1).

APPENDIX

In the above considerations we examined the momentum distribution function in states of definite energies and came to a coincidence of the classical results and those of the nonstatic SE in the case of small momenta, contrary to the implications of the static SE. It would certainly be interesting to try to find some explanation of this "nonphysical" result about momentum distributions of the nonstatic SE. In the case of potentials $U(x)=\text{const}\cdot x$ it is possible to obtain a detailed picture of the variation of the momentum p of the particles with t when the initial momentum p_0 at the moment $t_i=0$ of inclusion of the potential field is known.

Denote the potential energy as $U(x)=-Fx$, $-\infty<x<\infty$, where F is the constant (along x) force with which the field acts on our particle. Assume that at the moment $t_i=0$ of inclusion of this U the quantum state is given by WF $\psi_i(x)=\exp[ip_0x/\hbar]$. This ψ_i corresponds to an ensemble of particles having the same momentum $p=p_0$ at moment $t_i=0$, the distribution along the x axis being homogeneous. We shall seek the solution of the SE

$$i\hbar\frac{\partial\psi}{\partial t}=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x)\right]\psi$$

(m being the mass of the particles) for $t\geqslant 0$ in the form

$$\psi(t,x)=\exp[ip(t)x/\hbar]g(t) \tag{A.1}$$

where $p(t)$ is the momentum (at moment t) of a classical particle the initial momentum of which is equal to p_0, i.e., $p(t)=p_0+Ft$. The function $g(t)$ which satisfies the initial condition $g(0)=1$ is obtained trivially and the final result is

$$\psi(t,x)=\exp[ip(t)x/\hbar]\exp[(p_0^2t+p_0Ft^2+F^2t^3/3)/2i\hbar m] \quad \text{(A.2)}$$

This function corresponds to a homogeneous distribution of the particles along x at any moment t and to a definite momentum $p_x=p(t)$ of any particle at any t, i.e., the momentum of the particles varies precisely according to the Newtonian law $\dot{p}=F$. This is true for all possible momenta, including small ones. Thus, at least in this specific case, the classical conduct of particles having small momenta is an inherent feature of the nonstatic SE in a detailed picture of momentum variation too. As was said above, this is not in agreement with the implications of the stationary SE which we accept as giving a correct quantum picture of the motion of microparticles.

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